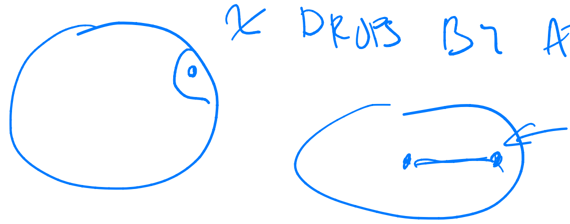
Lecture 2 - Critical Graphs IV

1 Double-critical graphs

A critical graph is *double-critical* if removing any pair of *adjacent* vertices decreases the chromatic number by two.

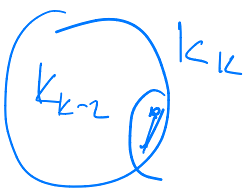
1: Show that a double-critical graph is critical.

χ DROPS BY AT MOST 1



DROP BY 2

2: Show that K_k is a double-critical graph.



$\chi(K_{k-2}) = \chi(K_k) - 2$

Lovász conjectured that K_k is the only one, and nowadays it is known under the name Double-Critical Graph Conjecture.

Conjecture 1 (Lovász)**.** *K_k is the only double-critical graph with chromatic number k.*

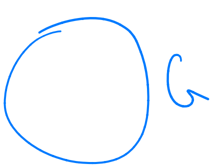
It is easy to verify that it holds for $k \leq 4$. Now, we give a proof for the case $k = 5$ which was found by Stiebitz in 1987. Since then no progress was done for any bigger k.

In the proof of the theorem bellow, we will use the following property of optimal colorings.

Observation 2. *Let c be an optimal coloring of a graph G. Then, for every color i, there is a i-colored vertex that is adjacent with a vertex of every other color.*

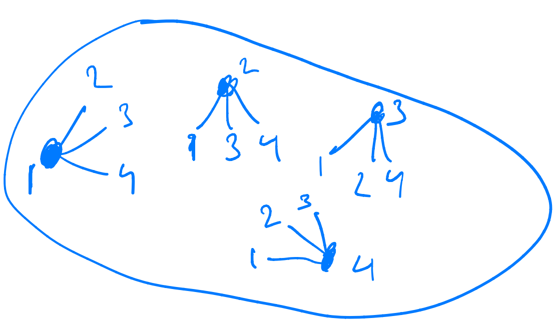
$\exists$ COLOR a S.T. NO VTX v $\chi(v) = a$ AND SEE ALL OTHER COLORS

3: Prove the observation.



COLORING WITH $\chi(G)$ COLORS

4-COLORING



$k = 4$

$a = 1$

RECOLOR ALL VERTICES COLORED a

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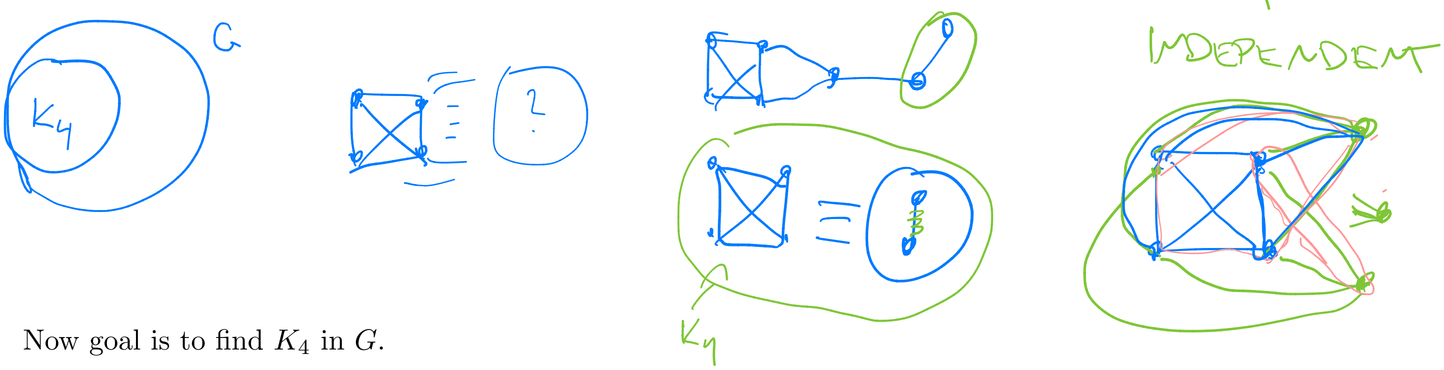
Theorem 3 (Stiebitz). *K_5 is the only double-critical graph with chromatic number* 5.

Proof. Let G be a double-critical graph with chromatic number 5.

4: Show that if G contains K_4, then G is K_5.

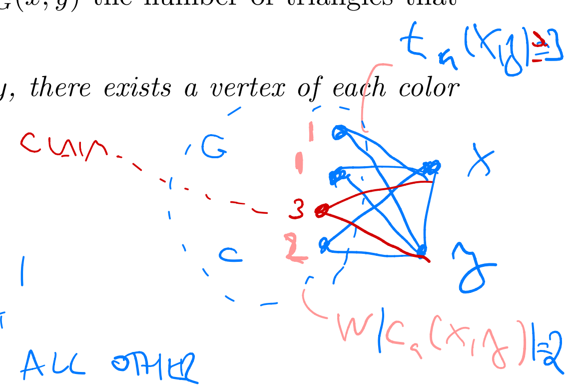
$G - K_4$ IS INDEPENDENT



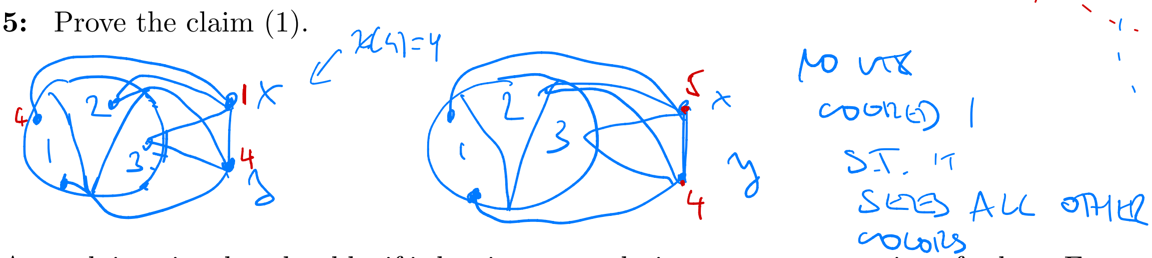
Now goal is to find K_4 in G.

Let $e = xy$ be any edge of G and c a 3-coloring of $G - x - y$. Denote by $c_G(x, y)$ the set of colors i of c for which exists i-colored vertex adjacent to both x and y. Also denote by $t_G(x, y)$ the number of triangles that contain the edge xy. Obviously, $|c_G(x, y)| \le t_G(x, y)$.

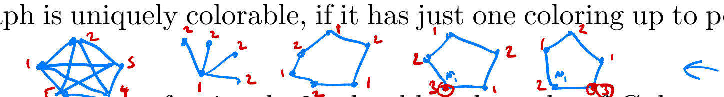
We claim first that (1) *for any edge xy and any 3-coloring c of $G - x - y$, there exists a vertex of each color adjacent to both x and y, i.e. $|c_G(x, y)| = 3$, and hence $t_G(x, y) \ge 3$.*



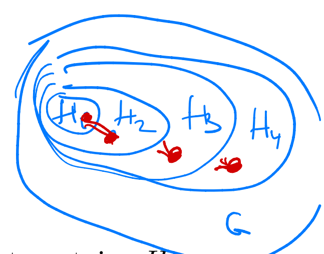
5: Prove the claim (1).



A graph is uniquely colorable, if it has just one coloring up to permutation of colors. Examples complete graphs, stars.



Take a sequence of uniquely 3-colorable subgraphs of G denoted $H_1, H_2, \ldots, H_r$ such that



(a) H_i is a uniquely 3-colorable graph on i vertices;

(b) H_i is a subgraph of H_{i+1};

(c) the sequence is maximal, i.e. there is no uniquely 3-colorable graph on $r + 1$ vertices that contains H_r.

$H_1 = K_1 =$ •

$H_2 = K_2 =$ •—•

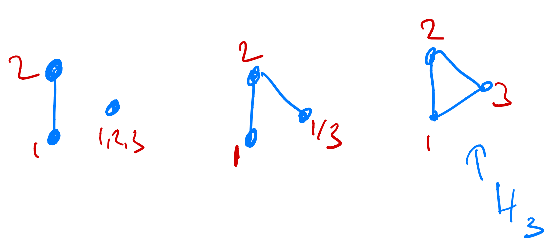
$\chi(H_1) = 1$ $\chi(H_2) = 2$

$\chi(H_3) = 3$ ....

6: Show that $r \ge 3$ and H_r is 3-chromatic.

$\chi(H_r) = 3$ FOR $r \ge 3$

$H_3 = K_3$?



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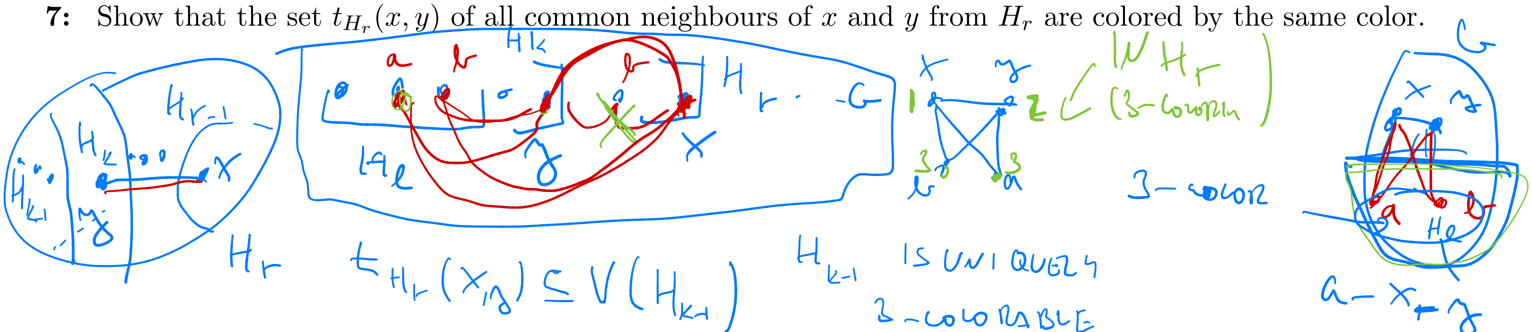
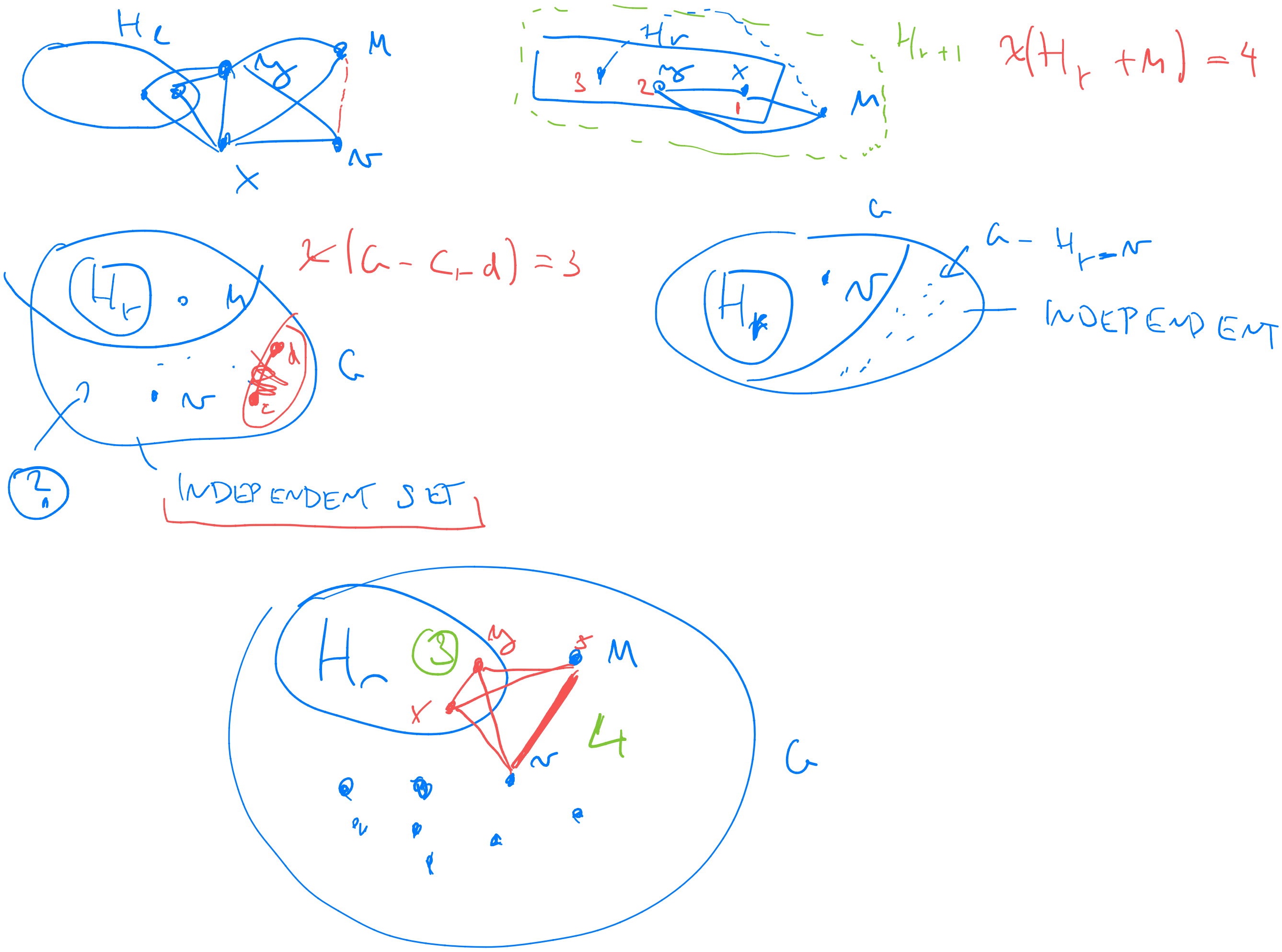
Let xy be the edge of H_r such that x is the vertex that is in H_r but not in H_{r-1}, and y is the neighbour of x that belongs to some H_k with biggest possible k. Consider some 3-coloring c of $G - x - y$.

7: Show that the set $t_{H_r}(x, y)$ of all common neighbours of x and y from H_r are colored by the same color.



Now, by the first claim, x and y have at least two common neighbours, say u and v, that do not belong to H_r (the one colored by other two colors). By the maximality of r, we have that $H_r + u$ and $H_r + v$ are 4-chromatic.

8: Finish the proof by showing that x, y, u, v for a clique.



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